

Supertraces on the Algebras of Observables of the Rational Calogero Model with Harmonic Potential

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Abstract

We define a complete set of supertraces on the algebra $SH_N(\nu)$, the algebra of observables of the N -body rational Calogero model with harmonic interaction. This result extends the previously known results for the simplest cases of $N = 1$ and $N = 2$ to arbitrary N . It is shown that $SH_N(\nu)$ admits $q(N)$ independent supertraces where $q(N)$ is a number of partitions of N into a sum of odd positive integers, so that $q(N) > 1$ for $N \geq 3$. Some consequences of the existence of several independent supertraces of $SH_N(\nu)$ are discussed such as the existence of ideals in associated W_∞ - type Lie superalgebras.

I Introduction

In this paper we investigate some properties of the associative algebras which were shown in [1, 2, 3] to underly the rational Calogero model [4] and were denoted as $SH_N(\nu)$ in [5]. Algebra $SH_N(\nu)$ is the associative algebra of polynomials constructed from arbitrary elements σ of the symmetric group S_N and the generating elements a_i^α obeying the following relations

$$\sigma a_i^\alpha = a_{\sigma(i)}^\alpha \sigma, \quad (1)$$

$$[a_i^\alpha, a_j^\beta] = \epsilon^{\alpha\beta} A_{ij}, \quad (2)$$

where $i, j = 1, \dots, N$, $\alpha, \beta = 0, 1$, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$, $\epsilon^{01} = 1$ and

$$A_{ij} = \delta_{ij} + \nu \tilde{A}_{ij}, \quad \tilde{A}_{ij} = \delta_{ij} \sum_{l=1}^N K_{il} - K_{ij}. \quad (3)$$

Here $K_{ij} \in S_N$ with $i, j = 1, \dots, N$, $i \neq j$, are the elementary permutations $i \leftrightarrow j$ satisfying the relations

$$K_{ij} = K_{ji}, \quad K_{ij} K_{ij} = 1, \quad K_{ij} K_{jl} = K_{jl} K_{li} = K_{li} K_{ij}$$

for $i \neq j \neq l \neq i$ and

$$K_{ij} K_{kl} = K_{kl} K_{ij}$$

if i, j, k, l are pairwise different. Note that in this paper repeated Latin indices i, j, k, \dots do not imply summation.

The defining relations (1)-(3) are consistent. In particular, the Jacobi identities

$$[a_i^\alpha, [a_j^\beta, a_k^\gamma]] + [a_j^\beta, [a_k^\gamma, a_i^\alpha]] + [a_k^\gamma, [a_i^\alpha, a_j^\beta]] = 0 \quad (4)$$

are satisfied.

An important property of $SH_N(\nu)$ which allows one to solve the Calogero model [4] is that this algebra possesses inner sl_2 automorphisms with the generators

$$T^{\alpha\beta} = \frac{1}{2} \sum_{i=1}^N (a_i^\alpha a_i^\beta + a_i^\beta a_i^\alpha), \quad (5)$$

$$[T^{\alpha\beta}, T^{\gamma\delta}] = \epsilon^{\alpha\gamma} T^{\beta\delta} + \epsilon^{\alpha\delta} T^{\beta\gamma} + \epsilon^{\beta\gamma} T^{\alpha\delta} + \epsilon^{\beta\delta} T^{\alpha\gamma}, \quad (6)$$

which act on the generating elements a_i^α as on sl_2 vectors

$$[T^{\alpha\beta}, a_i^\gamma] = \epsilon^{\alpha\gamma} a_i^\beta + \epsilon^{\beta\gamma} a_i^\alpha. \quad (7)$$

Calogero Hamiltonian is identified with the Cartan element T^{01} which according to (7) induces Z - gradation of $SH_N(\nu)$. The latter property allows one [2] to construct wave functions via the standard Fock procedure with the Fock vacuum $|0\rangle$ such that $a_i^0|0\rangle=0$. Thus, the elements a_i^α serve as generalized oscillators underlying the Calogero problem. The concrete realization of these oscillators in terms of Dunkl differential-difference operators [6] was presented in [1, 2].

These properties characterize the algebra $SH_N(\nu)$ as a natural generalization of the Heisenberg-Weyl algebra, the associative algebra of harmonic oscillators. Since the Lie algebra of quantum operators in the harmonic oscillator problem can be identified with the $W_{1+\infty}$ algebra [7], the Lie (super)algebras constructed from $SH_N(\nu)$ with the aid of supercommutators give rise to a class of the $W_{1+\infty}$ - type algebras which have been denoted as $W_{N,\infty}(\nu)$ in [8] where it was shown that all these algebras contain the Virasoro algebra as a subalgebra. The latter observation indicates that the algebras $SH_N(\nu)$ and $W_{N,\infty}(\nu)$ can be related to conformal models as well as to other classes of models in the range from quantum Hall effect [9] till higher-spin gauge theories [10] and KP hierarchy [11] where W_∞ - type algebras prove to be important. An additional argument in favor of the relationship of $SH_N(\nu)$ to the quantum Hall effect is due to the known fact that the Calogero model can be interpreted as a one-dimensional reduction of the full anyonic problem [12, 3].

For lower values of N , a nature of $SH_N(\nu)$ is rather well understood. $SH_1(\nu)$ is the ordinary Heisenberg-Weyl algebra (since ν - dependence is artificial in this case we will use

the notation SH_1). Properties of this algebra are very well studied (see *e.g.* [13]). Note that since the center of mass coordinates $1/N \sum_{i=1}^N a_i^\alpha$ decouple from everything else in the defining relations (1)-(3), the associative algebra $SH_N(\nu)$ has the structure $SH_N(\nu) = SH_1 \otimes SH'_N(\nu)$ where, by definition, $SH'_N(\nu)$ is the algebra of elements depending only on the relative coordinates $a_i^\alpha - a_j^\alpha$.

The properties of $SH'_2(\nu)$ are well studied too [14]. The algebra $SH'_2(\nu)$ is defined by the relations

$$[a^\alpha, a^\beta] = \epsilon^{\alpha\beta}(1 + 2\nu K), \quad (8)$$

where K is the only nontrivial element of S_2 while a^α are the relative motion oscillators. For the particular case of $\nu = 0$ one recovers the algebra SH_1 in the sector of the K independent elements.

In [14] it was shown that $SH'_2(\nu)$ admits a unique supertrace operation defined by the simple formula

$$str(1) = 1, \quad str(K) = -2\nu, \quad str(W) = str(WK) = 0 \quad (9)$$

for any polynomial $W \in SH'_2$ of the form

$$W = \sum_{n=1}^{\infty} W_{\alpha_1 \dots \alpha_n} a^{\alpha_1} \dots a^{\alpha_n} \quad (10)$$

with arbitrary totally symmetric multispinors $W_{\alpha_1 \dots \alpha_n}$. For the particular case of $\nu = 0$ one recovers the supertrace on SH_1 .

Furthermore it was shown in [14] by explicit evaluation of the invariant bilinear form $B(x, y) \stackrel{def}{=} str(xy)$ that for $\nu = l + \frac{1}{2}$ (l is any integer) $SH'_2(\nu)$ reduces to finite-dimensional matrix algebras up to some infinite-dimensional ideals \mathcal{I} which decouple from everything under the supertrace operation (9), *i.e.* $str(xy) = 0, \forall x \in \mathcal{I}$.

In [15] it was then observed that $SH'_2(\nu)$ is isomorphic to the factor of the enveloping algebra $U(osp(1; 2))$ of $osp(1; 2)$ over its ideal generated by the quadratic Casimir operator C_2 by factoring out all elements of the form $(C_2 - c_2)U(osp(1; 2))$ where $c_2 = \frac{1}{16}(4\nu^2 - 1)$ is an arbitrary constant. In its turn this observation clarified the origin of the ideals of $SH'_2(\nu)$ at $\nu = l + \frac{1}{2}$ as corresponding to the finite-dimensional representations of $osp(1; 2)$.

Although the algebra $SH_N(\nu)$ is getting interesting applications for any N , till now understanding of its algebraic properties for $N > 2$ is far from being satisfactory. In particular there is no interpretation of $SH_N(\nu)$ in terms of enveloping algebras of finite-dimensional superalgebras and nothing is known about ideals of $SH_N(\nu)$ which information is very important in applications.

In this paper we analyze the existence of the invariant supertrace operation on $SH_N(\nu)$ *i.e.* such a complex valued linear function $str(f)$ on $SH_N(\nu)$ that

$$str([f, g]) = 0, \quad \forall f, g \in SH_N(\nu) \quad (11)$$

with the convention that

$$[f, g] = fg - (-1)^{\pi(f)\pi(g)} gf, \quad (12)$$

where the parity π in $SH_N(\nu)$ is defined as follows:

$$\pi(a_i^\alpha) = 1, \quad \pi(K_{ij}) = 0. \quad (13)$$

Let us note that an attempt to define differently graded traces like, *e.g.*, an ordinary trace ($\pi \equiv 0$) unlikely leads to interesting results.

Knowledge of the supertrace operations on $SH_N(\nu)$ is useful in various respects. One of the most important applications of the supertrace is that it gives rise to n -linear invariant forms

$$str(a_1 a_2 \dots a_n) \tag{14}$$

that allows one to work with the algebra essentially in the same way as with the ordinary finite-dimensional matrix algebras and, for example, construct Lagrangians when working with dynamical theories based on $SH_N(\nu)$. Another useful property is that since null vectors of any invariant bilinear form span a both-side ideal of the algebra, this gives a powerful device for investigating ideals which decouple from everything under the supertrace operation as it happens in $SH_2(\nu)$ for half-integer ν . It is also worth mentioning that having an explicit form of the trilinear form in one or another basis is practically equivalent to defining a star-product law in the algebra.

An important motivation for the analysis of the supertraces of $SH_N(\nu)$ is due to its deep relationship with the analysis of the representations of this algebra, which in its turn gets applications to the analysis of the wave functions of the Calogero model. For example, given representation of $SH_N(\nu)$, one can speculate that it induces some supertrace on this algebra as (appropriately regularized) supertrace of (infinite) representation matrices. When the corresponding bilinear form degenerates this would imply that the representation becomes reducible.

As we show, the situation for $SH_N(\nu)$ is very interesting since starting from $N = 3$ it admits more than one independent supertrace in contrast to the cases of $N = 1$ and $N = 2$. This fact is in agreement with the results of [5] where it was shown that there exist many inequivalent lowest-weight type representations of $SH_N(\nu)$ for higher N (these representations are classified according to the representations of S_N .) Another important consequence of this phenomenon is that the Lie superalgebras $W_{N,\infty}(\nu)$ are not simple while appropriate their simple subalgebras possess non-trivial outer automorphisms.

The paper is organized as follows. In Section II we analyze consequences of S_N and sl_2 automorphisms of $SH_N(\nu)$. In Section III we discuss general properties of the supertraces and consequences of the existence of several independent supertraces. In Section IV we study the restrictions on supertraces of the group algebra of S_N considered as a subalgebra of $SH_N(\nu)$, which follow from the defining relations of $SH_N(\nu)$. These restrictions are called ground level conditions (*GLC*). They play a fundamental role in the problem since as we show in Section V every solution of *GLC* admits a unique extension to some supertrace on $SH_N(\nu)$. In Appendix A it is shown that the number of independent supertraces on $SH_N(\nu)$ equals to the number of partitions of N into a sum of odd positive integers. Some technical details of the proof of Section V are collected in Appendices B and C.

II Finite-Dimensional Groups of Automorphisms

The group algebra of S_N is the finite-dimensional subalgebra of $SH_N(\nu)$. The elements $\sigma \in S_N$ induce inner automorphisms of $SH_N(\nu)$. It is well known, that any $\sigma \in S_N$ can

be expanded into a product of pairwise commuting cycles

$$\sigma = c_1 c_2 c_3 \dots c_t, \quad (15)$$

where $c_{\mathfrak{w}}$, $\mathfrak{w} = 1, \dots, t$, are cyclic permutations acting on distinct subsets of values of indices i . For example, a cycle which acts on the first s indices as $1 \rightarrow 2 \rightarrow \dots \rightarrow s \rightarrow 1$ has the form

$$c = K_{12} K_{23} \dots K_{(s-1)s}. \quad (16)$$

We use the notation $|c|$ for the length of the cycle c . For the cycle (16), $|c| = s$. We take a convention that the cycles of unit length are associated with all values of i such that $\sigma(i) = i$, so that the relation $\sum_{\mathfrak{w}} |c_{\mathfrak{w}}| = N$ is true.

Given permutation $\sigma \in S_N$, we introduce a new set of basis elements $\mathfrak{B}_\sigma = \{b^I\}$ instead of $\{a_i^\alpha\}$ in the following way. For every cycle $c_{\mathfrak{w}}$ in the decomposition (15) ($\mathfrak{w} = 1, \dots, t$), let us fix some index $l_{\mathfrak{w}}$, which belongs to the subset associated with the cycle $c_{\mathfrak{w}}$. The basis elements $b_{\mathfrak{w}j}^\alpha$, $j = 1, \dots, |c_{\mathfrak{w}}|$, which realize 1-dimensional representations of the commutative cyclic group generated by $c_{\mathfrak{w}}$, have the form

$$b_{\mathfrak{w}j}^\alpha = \frac{1}{\sqrt{|c_{\mathfrak{w}}|}} \sum_{k=1}^{|c_{\mathfrak{w}}|} (\lambda_{\mathfrak{w}})^{jk} a_{l(\mathfrak{w},k)}^\alpha, \quad (17)$$

where $l(\mathfrak{w}, k) = c_{\mathfrak{w}}^{-k}(l_{\mathfrak{w}})$ and

$$\lambda_{\mathfrak{w}} = \exp(2\pi i / |c_{\mathfrak{w}}|). \quad (18)$$

From the definition (17) it follows that

$$c_{\mathfrak{w}} b_{\mathfrak{w}j}^\alpha = (\lambda_{\mathfrak{w}})^j b_{\mathfrak{w}j}^\alpha c_{\mathfrak{w}}, \quad (19)$$

$$c_{\mathfrak{w}} b_{\mathfrak{n}j}^\alpha = b_{\mathfrak{n}j}^\alpha c_{\mathfrak{w}}, \text{ for } \mathfrak{n} \neq \mathfrak{w} \quad (20)$$

and therefore

$$\sigma b_{\mathfrak{w}j}^\alpha = (\lambda_{\mathfrak{w}})^j b_{\mathfrak{w}j}^\alpha \sigma. \quad (21)$$

In what follows, instead of writing $b_{\mathfrak{w}j}^\alpha$ we use the notation b^I with the label I accounting for the full information about the index α , the index \mathfrak{w} enumerating cycles in (15), and the index j which enumerates various elements $b_{\mathfrak{w}j}^\alpha$ related to the cycle $c_{\mathfrak{w}}$, *i.e.* I ($I = 1, \dots, 2N$) enumerates all possible triples $\{\alpha, \mathfrak{w}, j\}$. We denote the index α , the cycle and the eigenvalue in (19) corresponding to some fixed index I as $\alpha(I)$, $c(I)$, and $\lambda_I = (\lambda_{\mathfrak{w}})^j$, respectively. The notation $\sigma(I) = \sigma_0$ implies that $b^I \in \mathfrak{B}_{\sigma_0}$. \mathfrak{B}_1 is the original basis of the generating elements a_i^α (here $\mathbf{1}$ is the unit permutation).

Let $\mathfrak{M}(\sigma)$ be the matrix which maps $\mathfrak{B}_1 \rightarrow \mathfrak{B}_\sigma$ in accordance with (17),

$$b^I = \sum_{i,\alpha} \mathfrak{M}_{i\alpha}^I(\sigma) a_i^\alpha. \quad (22)$$

Obviously this mapping is invertible. Using the matrix notations one can rewrite (21) as

$$\sigma b^I \sigma^{-1} = \sum_{J=1}^{2N} \Lambda_J^I(\sigma) b^J, \quad \forall b^I \in \mathfrak{B}_\sigma, \quad (23)$$

where $\Lambda_I^J(\sigma) = \delta_I^J \lambda_I$.

Every polynomial in $SH_N(\nu)$ can be expanded into a sum of monomials of the form

$$b^{I_1} b^{I_2} \dots b^{I_s} \sigma, \quad (24)$$

where all $\sigma(I_k) = \sigma$. Every monomial of this form realizes some one-dimensional representation of the Abelian group generated by all cycles $c_{\mathbf{w}}$ in the decomposition (15).

The commutation relations for the generating elements b^I follow from (2) and (3)

$$[b^I, b^J] = F^{IJ} = \mathcal{C}^{IJ} + \nu f^{IJ}, \quad (25)$$

where

$$\mathcal{C}^{IJ} = \epsilon^{\alpha(I)\alpha(J)} \delta_{c(I)c(J)} \delta_{\lambda_I \lambda_J^{-1}} \quad (26)$$

and

$$f^{IJ} = \sum_{i,j,\alpha,\beta} \mathfrak{M}_{i\alpha}^I(\sigma) \mathfrak{M}_{j\beta}^J(\sigma) \epsilon^{\alpha\beta} \tilde{A}_{ij}. \quad (27)$$

The indices I, J are raised and lowered with the aid of the symplectic form \mathcal{C}^{IJ}

$$\mu^I = \sum_J \mathcal{C}^{IJ} \mu_J, \quad \mu_I = \sum_J \mu^J \mathcal{C}_{JI}; \quad \sum_M \mathcal{C}_{IM} \mathcal{C}^{MJ} = -\delta_I^J. \quad (28)$$

Note that the elements b^I are normalized in (17) in such a way that the ν -independent part in (25) has the form (26).

Another important finite-dimensional algebra of inner automorphisms of $SH_N(\nu)$ is the sl_2 algebra which acts on the indices α . It is spanned by the S_N -invariant second-order polynomials (5). Evidently, $SH_N(\nu)$ decomposes into the infinite direct sum of only finite-dimensional irreducible representations of this sl_2 spanned by various homogeneous polynomials (24).

From the defining relations (1)-(3) it follows that $SH_N(\nu)$ is Z_2 - graded with respect to the automorphism

$$f(a_j^\alpha) = -a_j^\alpha, \quad f(K_{ij}) = K_{ij} \quad (29)$$

which gives rise to the parity π (13). In applications to higher-spin models, this automorphism distinguishes between bosons and fermions.

The algebra $SH_N(\nu)$ admits the antiautomorphism ρ ,

$$\rho(a_k^\alpha) = i a_k^\alpha, \quad \rho(K_{ij}) = K_{ij}, \quad (30)$$

which leaves invariant the basic relations (1)-(3) provided that an order of operators is reversed according to the defining property of antiautomorphisms: $\rho(AB) = \rho(B)\rho(A)$. From (15), (16) and (21) it follows that

$$\rho(\sigma) = \sigma^{-1}, \quad \rho(b^I) = i b^J, \quad (31)$$

where J is related to I in such a way that $\alpha(J) = \alpha(I)$, $\sigma(J) = (\sigma(I))^{-1}$, $c(J) = (c(I))^{-1}$ and $\lambda_J = \lambda_I^{-1}$. Note that in higher-spin theories the counterpart of ρ distinguishes between odd and even spins [16].

III General Properties of Supertrace

In this section we summarize some general properties to be respected by any supertrace in $SH_N(\nu)$.

Let A be an arbitrary associative Z_2 graded algebra with the parity function $\pi(x) = 0$ or 1. Suppose that A admits some supertrace operations str_p where the label p enumerates different nontrivial supertraces. We call a supertrace str even (odd) if $str(x) = 0 \forall x \in A$ such that $\pi(x) = 1(0)$. Let T_A be a linear space of supertraces on A . We say that $\dim T_A$ is the number of supertraces on A .

Given parity-preserving (anti)automorphism τ and supertrace operation str on A , $str(\tau(x))$ is some supertrace as well. For inner automorphisms τ ($\tau(x) = p x p^{-1}$, $\pi(p) = 0$) it follows from the defining property of the supertrace that $str(\tau(x)) = str(x)$. Thus, T_A forms a representation of the factor-group of the parity preserving automorphisms and antiautomorphisms of A over the normal subgroup of the inner automorphisms of A . Applying this fact to the original parity automorphism $(-1)^\pi$ one concludes that T_A can always be decomposed into a direct sum of subspaces of even and odd supertraces, $T_A = T_A^0 \oplus T_A^1$ and that $T_A^1 = 0$ if the parity automorphism is inner.

In the sequel we only consider the case where $\dim T_A < \infty$ and there are no nontrivial odd supertraces. Let $A = A_1 \otimes A_2$ with the associative algebras A_1 and A_2 endowed with some even supertrace operations t_1 and t_2 , respectively. The supertrace on A can be defined by setting $str(a_1 \otimes a_2) = t_1(a_1) t_2(a_2)$, $\forall a_1 \in A_1, \forall a_2 \in A_2$. As a result, one concludes that $T_A = T_{A_1} \otimes T_{A_2}$. In the case of $SH_N(\nu)$ one thus can always separate out a contribution of the center of mass coordinates as an overall factor (SH_1 admits the unique supertrace).

If A is finite-dimensional then the existence of two different supertraces indicates that A admits non-trivial both-side ideals. Actually, consider the bilinear form $B(f, g) = \alpha_1 str_1(fg) + \alpha_2 str_2(fg)$ with arbitrary parameters $\alpha_1, \alpha_2 \in \mathbb{C}$ and elements $f, g \in A$. The determinant of this bilinear form is some polynomial of α_1 and α_2 . Therefore it vanishes for certain ratios α_1/α_2 or α_2/α_1 according to the central theorem of algebra. Thus, for these values of the parameters the bilinear form B degenerates and admits non-trivial null vectors x , $B(x, g) = 0, \forall g \in A$. It is easy to see that the linear space \mathcal{I} of all null vectors x is some both-side ideal of A . For infinite-dimensional algebras the existence of several supertraces does not necessarily imply the existence of ideals. As mentioned in introduction the existence of several supertrace operations may be related to the existence of inequivalent representations. Also it is worth mentioning that for the case of infinite-dimensional algebras and representations under investigation it can be difficult to use the standard (*i.e.* matrixwise) definition of the supertrace. In this situation the formal definition of the supertraces on the algebra we implement in this paper is the only rigorous one.

Let l_A be the Lie superalgebra which is isomorphic to A as a linear space and is endowed with the product law (12). It contains the subalgebra $sl_A \in l_A$ spanned by elements g such that $str_p(g) = 0$ for all p . Evidently sl_A forms the ideal of l_A . The factor algebra $t_A = l_A / sl_A$ is a commutative Lie algebra isomorphic to T_A^* as a linear space. Elements of t_A different from the unit element of A (which exist if $\dim T_A > 1$) can induce outer automorphisms of sl_A . Let us note that it is this sl_A Lie superalgebra which usually has

physical applications. For the case of $SH_N(\nu)$ under consideration the algebra $l_{SH_N(\nu)}$ is identified with the algebra $W_{N,\infty}(\nu)$ introduced in [8]. We therefore conclude that these algebras are not simple for $N > 2$ because it is shown below that $SH_N(\nu)$ admits several supertraces for $N > 2$. Instead one can consider the algebras $sW_{N,\infty}(\nu)$.

Let l_A contain some subalgebra \mathcal{L} such that A decomposes into a direct sum of irreducible representations of \mathcal{L} with respect to the adjoint action of \mathcal{L} on A via supercommutators. Then, only trivial representations of \mathcal{L} can contribute to any supertrace on A . Actually, consider some non-trivial irreducible representation R of \mathcal{L} . Any $r \in R$ can be represented as

$$r = \sum_j [l_j, r_j], \quad l_j \in \mathcal{L}, \quad r_j \in R \quad (32)$$

since elements of the form (32) span the invariant subspace in R . From (11) it follows then that $str(r) = 0, \forall r \in R$.

From the definition of the supertrace it follows that

$$str(a_1 a_2) + str(a_2 a_1) = 0 \quad (33)$$

for arbitrary odd elements a_1 and a_2 of A . A simple consequence of this relation is that

$$str(a_1 a_2 \dots a_n + a_2 \dots a_n a_1 + \dots + a_n a_1 \dots a_{n-1}) = 0 \quad (34)$$

is true for an arbitrary even n if all a_i are some odd elements of A . Since we assume that the supertrace is even (34) is true for any n . This simple property turns out to be practically useful because, when odd generating elements are subject to some commutation relations with the right hand sides expressed via even generating elements like in (2), it often allows one to reduce evaluation of the supertrace of a degree- n polynomial of a_i to supertraces of lower degree polynomials.

Another useful property is that in order to show that the characteristic property of the supertrace (11) is true for any $x, g \in A$, it suffices to show this for a particular case where x is arbitrary while g is an arbitrary generating element of some fixed system of generating elements. Then (11) for general x and g will follow from the properties that A is associative and str is linear. For the particular case of $SH_N(\nu)$ this means that it is enough to set either $g = a_i^\alpha$ or $g = K_{ij}$.

Let us now turn to some specific properties of $SH_N(\nu)$ as a particular realization of A .

By identifying \mathcal{L} with sl_2 (5) and taking into account that $SH_N(\nu)$ decomposes into a direct sum of irreducible finite-dimensional representations of sl_2 , one arrives at the following

Lemma 1: *$str(x)$ can be different from zero only when x is sl_2 -singlet, i.e. $[T^{\alpha\beta}, x] = 0$.*

Corollary: Any supertrace on $SH_N(\nu)$ is even.

Analogously one deduces consequences of the S_N symmetry. In particular, one proves

Lemma 2: *Given $c \in S_N$ such that $cF = \mu Fc$ for some element F and any constant $\mu \neq 1$, $str(F) = 0$. Given monomial $F = b^{I_1} b^{I_2} \dots b^{I_s} \sigma$ with $b^{I_k} \in \mathfrak{B}_\sigma$ and a cycle c_0 in the decomposition (15) of σ one concludes that $str(F) = 0$ if $\prod_{k: c(I_k)=c_0} \lambda_{I_k} \neq 1$ where λ_{I_k} are the eigenvalues (21) of b^{I_k} .*

IV Ground Level Conditions

Let us analyze restrictions on a form of $str(a)$, $a \in S_N$, which follow from the defining relations of $SH_N(\nu)$.

Firstly, we describe supertraces on the group algebra of S_N . Let some permutation σ decomposes into n_1 cycles of length 1, n_2 cycles of length 2, ... and n_N cycles of length N . The non-negative integers n_k satisfy the relation

$$\sum_{k=1}^N kn_k = N \quad (35)$$

and fix σ up to some conjugation $\sigma \rightarrow \tau\sigma\tau^{-1}$, $\tau \in S_N$. Thus

$$str(\sigma) = \varphi(n_1, n_2, \dots, n_N), \quad (36)$$

where $\varphi(n_1, n_2, \dots, n_N)$ is an arbitrary function. Obviously the linear space of invariant functions on S_N (*i.e.* such that $f(\tau\sigma\tau^{-1}) = f(\sigma)$) coincides with the linear space of supertraces on the group algebra of S_N . Therefore, the dimension of the linear space of supertraces is equal to the number $p(N)$ of independent solutions of (35), the number of conjugacy classes of S_N . One can introduce the generating function for $p(N)$ as $P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)}$. The properties of this generating function and of the quantities $p(N)$ are discussed in details *e.g.* in [17].

According to the general argument of the previous section, the existence of several independent traces implies that the group algebra of S_N must have some ideals. Indeed it can be shown to decompose into a direct sum of matrix algebras Mat_n .

Since the group algebra of S_N is embedded into $SH_N(\nu)$ some additional restrictions on the functions $\varphi(n_1, n_2, \dots, n_N)$ follow from (11) and the defining relations (2)-(3) of $SH_N(\nu)$. Actually, consider some elements b^I such that $\lambda_I = -1$. Then, one finds from (11) and (21) that $str(b^I b^J \sigma) = -str(b^J \sigma b^I) = str(b^J b^I \sigma)$ and therefore

$$str([b^I, b^J]\sigma) = 0. \quad (37)$$

Since these conditions restrict supertraces of degree-0 polynomials of a_i^α we call them ground level conditions (*GLC*). Thus for every permutation σ and any even integer $2k$ such that there exists some cycle c of length $|c| = 2k$ in the decomposition (15) we have *GLC* (37) with b^I such that $c(I) = c$. Note however that if $\lambda_J \neq -1$ or $c(J) \neq c(I)$ then the relation $str([b^I, b^J]\sigma) = 0$ is trivially satisfied as a consequence of *Lemma 2*.

It is convenient to rewrite *GLC* in the following form

$$str(c_0 \sigma_0) = -str\left(\left([b_{0k}^0, b_{0k}^1] - 1\right)c_0 \sigma_0\right), \quad (38)$$

where c_0 is any cycle of even length $2k$ in the decomposition of the permutation $\sigma = c_0 \sigma_0$ and b_{0k}^α is the corresponding variable (17) with $(\lambda_0)^k = -1$, *i.e.* $c_0 b_{0k}^\alpha = -b_{0k}^\alpha c_0$, $\sigma_0 b_{0k}^\alpha = b_{0k}^\alpha \sigma_0$ and $\lambda_0 = \exp(2\pi i/|c_0|)$.

To work out the explicit form of the restrictions on the functions $\varphi(n_1, n_2, \dots, n_N)$ which follow from *GLC* one has to use the following simple facts from the theory of the symmetric group:

Lemma 3: Let c_1 and c_2 be two distinct cycles in the decomposition (15). Let indices i_1 and i_2 belong to the subsets of indices associated with the cycles c_1 and c_2 , respectively. Then the permutation $c = c_1 c_2 K_{i_1 i_2}$ is a cycle of length $|c| = |c_1| + |c_2|$.

Lemma 4: Given cyclic permutation $c \in S_N$, let $i \neq j$ be two indices such that $c^k(i) = j$, where k is some positive integer, $k < |c|$. Then $cK_{ij} = c_1 c_2$ where $c_{1,2}$ are some non-coinciding mutually commuting cycles such that $|c_1| = k$ and $|c_2| = |c| - k$.

Using the definition (17), the commutation relations (1)-(3) and Lemmas 3 and 4 one reduces *GLC* to the following system of equations:

$$\begin{aligned}
& n_{2k} \varphi(n_1, \dots, n_{2k}, \dots, n_N) \\
&= -\nu n_{2k} \left(2 \sum_{s \neq k, s=1}^{2k-1} O_s \varphi(n_1, \dots, n_s + 1, \dots, n_{2k-s} + 1, \dots, n_{2k} - 1, \dots, n_N) \right. \\
&+ 2O_k \varphi(n_1, \dots, n_k + 2, \dots, n_{2k} - 1, \dots, n_N) \\
&+ \sum_{s \neq 2k; s=1}^N s n_s \varphi(n_1, \dots, n_s - 1, \dots, n_{2k} - 1, \dots, n_{2k+s} + 1, \dots, n_N) \\
&\left. + 2k(n_{2k} - 1) \varphi(n_1, \dots, n_{2k} - 2, \dots, n_{4k} + 1, \dots, n_N) \right) \tag{39}
\end{aligned}$$

where $O_k = 0$ for k even and $O_k = 1$ for k odd.

Let us note that by virtue of the substitution

$$\varphi(n_1, \dots, n_N) = \nu^{E(\sigma)} \tilde{\varphi}(n_1, \dots, n_N), \tag{40}$$

where $E(\sigma)$ is the number of cycles of even length in the decomposition of σ (15), *i.e.*

$$E(\sigma) = n_2 + n_4 + \dots \tag{41}$$

one can get rid of the explicit dependence of ν from *GLC* (39). As a result, there are two distinguishing cases, $\nu = 0$ and $\nu \neq 0$.

For lower N the conditions (39) take the form

$$\varphi(0, 1) + 2\nu \varphi(2, 0) = 0 \tag{42}$$

for $N = 2$ (*cf.* (9)),

$$\varphi(1, 1, 0) + 2\nu \varphi(3, 0, 0) + \nu \varphi(0, 0, 1) = 0 \tag{43}$$

for $N = 3$ and

$$\begin{aligned}
\varphi(2, 1, 0, 0) + 2\nu \varphi(4, 0, 0, 0) + 2\nu \varphi(1, 0, 1, 0) &= 0 \\
\varphi(0, 2, 0, 0) + 2\nu \varphi(2, 1, 0, 0) + 2\nu \varphi(0, 0, 0, 1) &= 0 \\
\varphi(0, 0, 0, 1) + 4\nu \varphi(1, 0, 1, 0) &= 0
\end{aligned}$$

for $N = 4$. As a result one finds 1-parametric families of solutions for $N = 1$ and $N = 2$ and 2-parametric families of solutions for $N = 3$ and $N = 4$.

Let G_N be the number of independent solutions of (39). As we show in the next section $G_N = \dim T_{SH_N(\nu)}$ for all ν . In other words all other conditions on the supertrace do not

impose any restrictions on the functions $\varphi(n_1, \dots, n_N)$ but merely express supertraces of higher order polynomials of a_i^α in terms of $\varphi(n_1, \dots, n_N)$.

In the Appendix A we prove the following

Theorem 1: $G_N = q(N)$ where $q(N)$ is a number of partitions of N into a sum of odd positive integers, i.e. the number of the solutions of the equation $\sum_{k=0}^{\infty} (2k+1)n_k = N$ for non-negative integers n_i .

One can guess this result from the particular case of $\nu = 0$ where *GLC* tell us that $\varphi(n_1, \dots, n_N)$ can be nonvanishing (and arbitrary) only when all $n_{2k} = 0$. Interestingly enough, G_N remains the same for $\nu \neq 0$.

V Supertrace for General Elements

In this section we prove

Theorem 2: $\dim T_{SH_N(\nu)} = G_N$ where G_N is the number of independent solutions of the ground level conditions (39).

The proof of the *Theorem 2* will be given in a constructive way by virtue of the following double induction procedure:

(i). Assuming that *GLC* are true and $\text{str}\{b^I, P_p(a)\sigma\} = 0 \ \forall P_p(a), \sigma$ and I provided that $b^I \in \mathfrak{B}_\sigma$ and

$$\begin{aligned} &\lambda(I) \neq -1; p \leq k \text{ or} \\ &\lambda(I) = -1, E(\sigma) \leq l, p \leq k \text{ or} \\ &\lambda(I) = -1; p \leq k-2, \end{aligned}$$

where $P_p(a)$ is an arbitrary degree p polynomial of a_i^α (p is odd) and $E(\sigma)$ is the number of cycles of even length in the decomposition (15) of σ , one proves that there exists such a unique extension of the supertrace that the same is true for $l \rightarrow l+1$.

(ii). Assuming that $\text{str}\{b^I, P_p(a)\sigma\} = 0 \ \forall P_p(a), \sigma$ and b^I such that $\sigma(I) = \sigma, p \leq k$ one proves that there exists such a unique extension of the supertrace that the assumption (i) is true for $k \rightarrow k+2$ and $l = 0$.

As a result this inductive procedure extends uniquely any solution of *GLC* to some supertrace on the whole $SH_N(\nu)$. (Let us remind ourselves that the supertrace of any odd element of $SH_N(\nu)$ is trivially zero by sl_2 invariance).

The inductive proof of the *Theorem 2* is based on the S_N covariance of the whole setting and the following important

Lemma 5: Given permutation σ which has $E(\sigma)$ cycles of even length in the decomposition (15), the quantity $f^{IJ}\sigma$ for $\sigma(I) = \sigma(J) = \sigma$ and $\lambda_I = \lambda_J = -1$ can be uniquely expanded as $f^{IJ}\sigma = \sum_q \alpha_q \sigma_q$ where α_q are some coefficients and $E(\sigma_q) = E(\sigma) - 1 \ \forall q$.

Lemma 5 is a simple consequence of the particular form of the structure coefficients f^{IJ} (27) and *Lemmas 3* and *4*. The proof is straightforward. Let us stress that it is *Lemma 5* which accounts for the specific properties of the algebra $SH_N(\nu)$ in the analysis of this section.

In practice it is convenient to work with the exponential generating functions

$$\Psi_\sigma(\mu) = \text{str} \left(e^S \sigma \right), \quad S = \sum_{L=1}^{2N} (\mu_L b^L), \quad (44)$$

where σ is some fixed element of S_N , $b^L \in \mathfrak{B}_\sigma$ and $\mu_L \in \mathbb{C}$ are independent parameters. By differentiating over μ_L one can obtain an arbitrary polynomial of b^L in front of σ . The exponential form of the generating functions implies that these polynomials are Weyl ordered. In these terms the induction on a degree of polynomials is equivalent to the induction on a degree of homogeneity in μ of the power series expansions of $\Psi_\sigma(\mu)$.

As a consequence of the general properties discussed in the preceding sections the generating function $\Psi_\sigma(\mu)$ must be invariant under the S_N similarity transformations

$$\Psi_{\tau\sigma\tau^{-1}}(\mu) = \Psi_\sigma(\tilde{\mu}), \quad (45)$$

where the S_N transformed parameters are of the form

$$\tilde{\mu}_I = \sum_J \left(\mathfrak{M}(\tau\sigma\tau^{-1}) \mathfrak{M}^{-1}(\tau) \Lambda^{-1}(\tau) \mathfrak{M}(\tau) \mathfrak{M}^{-1}(\sigma) \right)_I^J \mu_J \quad (46)$$

and matrices $\mathfrak{M}(\sigma)$ and $\Lambda(\sigma)$ are defined in (22) and (23).

In accordance with the general argument of Section III the necessary and sufficient conditions for the existence of even supertrace are the S_N -covariance conditions (45) and the condition that

$$\text{str} \left\{ b^L, (\exp S) \sigma \right\} = 0 \quad \text{for any } \sigma \text{ and } L. \quad (47)$$

To transform (47) to an appropriate form, let us use the following two general relations which are true for arbitrary operators X and Y and the parameter $\mu \in \mathbb{C}$:

$$X \exp(Y + \mu X) = \frac{\partial}{\partial \mu} \exp(Y + \mu X) + \int t_2 \exp(t_1(Y + \mu X)) [X, Y] \exp(t_2(Y + \mu X)) D^1 t, \quad (48)$$

$$\exp(Y + \mu X) X = \frac{\partial}{\partial \mu} \exp(Y + \mu X) - \int t_1 \exp(t_1(Y + \mu X)) [X, Y] \exp(t_2(Y + \mu X)) D^1 t \quad (49)$$

with the convention that

$$D^{n-1} t = \delta(t_1 + \dots + t_n - 1) \theta(t_1) \dots \theta(t_n) dt_1 \dots dt_n. \quad (50)$$

The relations (48) and (49) can be derived with the aid of the partial integration (*e.g.* over t_1) and the following formula

$$\frac{\partial}{\partial \mu} \exp(Y + \mu X) = \int \exp(t_1(Y + \mu X)) X \exp(t_2(Y + \mu X)) D^1 t, \quad (51)$$

which can be proven by expanding in power series. The well-known formula

$$[X, \exp(Y)] = \int \exp(t_1 Y) [X, Y] \exp(t_2 Y) D^1 t \quad (52)$$

is a consequence of (48) and (49).

With the aid of (48), (49) and (21) one rewrites (47) as

$$(1 + \lambda_L) \frac{\partial}{\partial \mu_L} \Psi_\sigma(\mu) = \int (\lambda_L t_1 - t_2) \text{str} \left(\exp(t_1 S) [b^L, S] \exp(t_2 S) \sigma \right) D^1 t. \quad (53)$$

This condition should be true for any σ and L and plays the central role in the analysis of this section.

There are two essentially distinguishing cases, $\lambda_L \neq -1$ and $\lambda_L = -1$. In the latter case, the equation (53) takes the form

$$0 = \int \text{str} \left(\exp(t_1 S) [b^L, S] \exp(t_2 S) \sigma \right) D^1 t, \quad \lambda_L = -1. \quad (54)$$

In Appendix B we show by induction that the equations (53) and (54) are consistent in the following sense

$$(1 + \lambda_K) \frac{\partial}{\partial \mu_K} \int (\lambda_L t_1 - t_2) \text{str} \left(\exp(t_1 S) [b^L, S] \exp(t_2 S) \sigma \right) D^1 t - (L \leftrightarrow K) = 0, \quad (55)$$

$$\lambda_L \neq -1, \lambda_K \neq -1$$

and

$$(1 + \lambda_K) \frac{\partial}{\partial \mu_K} \int \text{str} \left(\exp(t_1 S) [b^L, S] \exp(t_2 S) \sigma \right) D^1 t = 0, \quad \lambda_L = -1. \quad (56)$$

Note that this part of the proof is quite general and does not depend on a concrete form of the commutation relations of a_i^α in (2).

By expanding the exponential e^S in (44) into power series in μ_K (equivalently b^K) one concludes that the equation (53) uniquely reconstructs the supertrace of monomials containing b^K with $\lambda_K \neq -1$ (from now on called regular polynomials) via supertraces of some lower order polynomials. The consistency conditions (55) and (56) then guarantee that (53) does not impose any additional conditions on the supertraces of lower degree polynomials and allow one to represent the generating function in the form

$$\begin{aligned} \Psi_\sigma &= \Phi_\sigma(\mu) \\ &+ \sum_{L: \lambda_L \neq -1} \int_0^1 \frac{\mu_L d\tau}{1 + \lambda_L} \int D^1 t (\lambda_L t_1 - t_2) \text{str} \left(e^{t_1(\tau S'' + S')} [b^L, (\tau S'' + S')] e^{t_2(\tau S'' + S')} \sigma \right), \end{aligned} \quad (57)$$

where we have introduced the generating functions Φ_σ for the supertrace of special polynomials, *i.e.* the polynomials depending only on b^L with $\lambda_L = -1$,

$$\Phi_\sigma(\mu) \stackrel{\text{def}}{=} \text{str} \left(e^{S'} \sigma \right) = \Psi_\sigma(\mu) \Big|_{(\mu_I = 0 \ \forall I: \lambda_I \neq -1)} \quad (58)$$

and

$$S' = \sum_{L: b^L \in \mathfrak{B}_\sigma, \lambda_L = -1} (\mu_L b^L); \quad S'' = S - S'. \quad (59)$$

The relation (57) successively expresses the supertrace of higher order regular polynomials via the supertraces of lower order polynomials.

One can see that the arguments above prove effectively the inductive hypotheses **(i)** and **(ii)** for the particular case where either the polynomials $P_p(a)$ are regular and/or $\lambda_I \neq -1$. Note that for this case the induction on the number of cycles of even length **(i)** is trivial: one simply proves that a power of polynomial can be increased by two.

Let us now turn to the less trivial case of the special polynomials:

$$\text{str} \left\{ b^I, (\exp S') \sigma \right\} = 0, \quad \lambda_I = -1. \quad (60)$$

Consider the part of $str \{b^I, (exp S')\sigma\}$ which is of order k in μ and suppose that $E(\sigma) = l + 1$. According to (54) the conditions (60) give

$$0 = \int str \left(exp(t_1 S') [b^I, S'] exp(t_2 S') \sigma \right) D^1 t. \quad (61)$$

Substituting $[b^I, S'] = \mu^I + \nu \sum_M f^{IM} \mu_M$, where the quantities f^{IJ} and μ^I are defined in (25)-(28), one can rewrite the equation (61) in the form

$$\mu^I \Phi_\sigma(\mu) = -\nu \int str \left(exp(t_1 S') \sum_M f^{IM} \mu_M exp(t_2 S') \sigma \right) D^1 t. \quad (62)$$

Now we use the inductive hypothesis (i). The right hand side of (62) is a supertrace of at most a degree $k - 1$ polynomial of a_i^α in the sector of degree k polynomials in μ . Therefore one can use the inductive hypothesis (i) to obtain

$$\int str \left(exp(t_1 S') \sum_M f^{IM} \mu_M exp(t_2 S') \sigma \right) D^1 t = \int str \left(exp(t_2 S') exp(t_1 S') \sum_M f^{IM} \mu_M \sigma \right) D^1 t,$$

where we made use of the simple fact that $str(S' F \sigma) = -str(F \sigma S') = str(F S' \sigma)$ due to the definition of S' .

As a result, the inductive hypothesis allows one to transform (60) to the following form

$$X^I \equiv \mu^I \Phi_\sigma(\mu) + \nu str \left(exp(S') \sum_M f^{IM} \mu_M \sigma \right) = 0. \quad (63)$$

By differentiating this equation with respect to μ_J one obtains after symmetrization

$$\frac{\partial}{\partial \mu_J} (\mu^I \Phi_\sigma(\mu)) + (I \leftrightarrow J) = -\nu \int str \left(e^{t_1 S'} b^J e^{t_2 S'} \sum_M f^{IM} \mu_M \sigma \right) D^1 t + (I \leftrightarrow J). \quad (64)$$

An important point is that the system of equations (64) is equivalent to the original equations (63) except for the ground level part $\Phi_\sigma(0)$. This can be easily seen from the simple fact that the general solution of the system of equations $\frac{\partial}{\partial \mu_J} X^I(\mu) + \frac{\partial}{\partial \mu_I} X^J(\mu) = 0$ for entire functions $X^I(\mu)$ is of the form $X^I(\mu) = X^I(0) + \sum_J c^{IJ} \mu_J$ where $X^I(0)$ and $c^{JI} = -c^{IJ}$ are some constants. The part of (63) linear in μ is however equivalent to the ground level conditions analyzed in the previous section. Thus (64) contains all information additional to (39). For this reason we will from now on analyze the equation (64).

Using again the inductive hypothesis we move b^I to the left and to the right with equal weights to get

$$\begin{aligned} \frac{\partial}{\partial \mu_J} \mu^I \Phi_\sigma(\mu) + (I \leftrightarrow J) &= -\frac{\nu}{2} \sum_M str \left(exp(S') \{b^J, f^{IM}\} \mu_M \sigma \right) \\ &- \frac{\nu}{2} \int \sum_{L,M} (t_1 - t_2) str \left(exp(t_1 S') F^{JL} \mu_L exp(t_2 S') f^{IM} \mu_M \sigma \right) D^1 t + (I \leftrightarrow J). \end{aligned} \quad (65)$$

The last term on the right hand side of this expression can be shown to vanish under the supertrace operation due to the factor of $(t_1 - t_2)$, so that one is left with the equation

$$L^{IJ} \Phi_\sigma(\mu) = -\frac{\nu}{2} R^{IJ}(\mu), \quad (66)$$

where

$$R^{IJ}(\mu) = \sum_M \text{str} \left(\exp(S') \{b^J, f^{IM}\} \mu_M \sigma \right) + (I \leftrightarrow J) \quad (67)$$

and

$$L^{IJ} = \frac{\partial}{\partial \mu_J} \mu^I + \frac{\partial}{\partial \mu_I} \mu^J. \quad (68)$$

The differential operators L^{IJ} satisfy the standard $sp(2E(\sigma))$ commutation relations

$$[L^{IJ}, L^{KL}] = - \left(C^{IK} L^{JL} + C^{IL} L^{JK} + C^{JK} L^{IL} + C^{JL} L^{IK} \right). \quad (69)$$

We show by induction in Appendix C that this algebra is consistent with the right-hand side of the basic relation (66) *i.e.* that

$$[L^{IJ}, R^{KL}] - [L^{KL}, R^{IJ}] = - \left(C^{IK} R^{JL} + C^{JL} R^{IK} + C^{JK} R^{IL} + C^{IL} R^{JK} \right). \quad (70)$$

Generally, these consistency conditions guarantee that the equations (66) express $\Phi_\sigma(\mu)$ in terms of R^{IJ} in the following way

$$\Phi_\sigma(\mu) = \Phi_\sigma(0) + \frac{\nu}{8E(\sigma)} \sum_{I,J=1}^{2E(\sigma)} \int_0^1 \frac{dt}{t} (1 - t^{2E(\sigma)}) (L_{IJ} R^{IJ})(t\mu), \quad (71)$$

provided that

$$R^{IJ}(0) = 0. \quad (72)$$

The latter condition must hold for the consistency of (66) since its left hand side vanishes at $\mu_I = 0$. In the formula (71) it guarantees that the integral on t converges. In the case under consideration the property (72) is indeed true as a consequence of the definition (67).

Taking into account *Lemma 5* and the explicit form of R^{IJ} (67) one concludes that the equation (71) expresses uniquely the supertrace of special polynomials via the supertraces of polynomials of lower degrees or via the supertraces of special polynomials of the same degree with a lower number of cycles of even length provided that the μ independent term $\Phi_\sigma(0)$ is an arbitrary solution of *GLC*. This completes the proof of *Theorem 2*.

Comment 1: *The formulae (57) and (71) can be effectively used in practical calculations of supertraces of particular elements of $SH_N(\nu)$.*

Comment 2: *Any supertrace on $SH_N(\nu)$ is determined unambiguously in terms of its values on the group algebra of S_N .*

Corollary: Any supertrace on $SH_N(\nu)$ is ρ -invariant, $\text{str}(\rho(x)) = \text{str}(x) \forall x \in SH_N(\nu)$, for the antiautomorphism ρ (30).

This is true due to the *Comment 2* because σ and $\sigma^{-1} = \rho(\sigma)$ belong to the same conjugacy class of S_N so that $\text{str}(\rho(\sigma)) = \text{str}(\sigma)$.

VI Conclusions.

In this paper we have shown that the algebras $SH_N(\nu)$ can be endowed with $q(N)$ independent supertrace operations where $q(N)$ is the number of partitions of N into a sum of odd positive integers. We hope to apply the supertraces constructed in this paper to the analysis of the invariant forms of $SH_N(\nu)$. Although a definition of the supertraces on $SH_N(\nu)$ behaves regularly with the parameter ν (in particular, the number of supertraces $q(N)$ is ν independent) one can expect that this is not the case for the related bilinear forms which can degenerate for some special values of ν thus giving rise to ideals of $SH_N(\nu)$ as it happens [14] for the simplest case of $N = 2$. The analysis of the structure of these ideals is a challenging problem important for various application of $SH_N(\nu)$, including analysis of its representations. We are going to study this problem for some lower values of $N > 2$ in the future publication.

In conclusion let us note that the method of the analysis of supertraces presented in this paper is rather general. Practically, the only information of the specific structure of $SH_N(\nu)$ is that *Lemma 5* is true. Hopefully one can use the analogous methods for the analysis of supertraces of other associative algebras.

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APPENDICES

A Independence G_N of ν .

For the case $\nu = 0$ it was argued in section IV that (39) possesses $q(N)$ independent solutions. Let us now consider the case $\nu \neq 0$. By induction on a number of cycles of even length $e = E(\sigma)$ we show that given σ with $E(\sigma) = e \geq 1$ there is only one independent equation on $str(\sigma)$ provided that all equations (39) with $E(\sigma) = e' < e$ are resolved. In this proof we set $\nu = 1$ that does not lead to the loss of generality due to the scaling property (40). The first step of the induction consists of the observation that there are no equations for the case $E(\sigma) = 0$.

Let us consider the case where there are two equations (38) on $str(\sigma)$ for some σ . This is only possible if $\sigma = c_1 c_2 \sigma'$ where c_1 and c_2 are some cycles in the decomposition of σ such that $|c_1| = 2k$, $|c_2| = 2l$ and $k \neq l$. Note that $E(\sigma') = E(\sigma) - 2 = e - 2$.

Without loss of generality let us set

$$c_1 = K_{12} K_{23} \dots K_{(2k-1)2k}, \quad c_2 = K_{(2k+1)(2k+2)} \dots K_{(2k+2l-1)(2k+2l)}, \quad (\text{A1})$$

$$b_1^\alpha = \frac{1}{\sqrt{2k}}(a_1^\alpha - a_2^\alpha + \dots - a_{2k}^\alpha), \quad b_2^\alpha = \frac{1}{\sqrt{2l}}(a_{2k+1}^\alpha - a_{2k+2}^\alpha + \dots - a_{2k+2l}^\alpha). \quad (\text{A2})$$

Also we introduce

$$c = K_{1(2k+1)}c_1c_2 = K_{12}K_{23} \dots K_{(2k+2l-1)(2k+2l)} \quad (\text{A3})$$

and

$$b^\alpha = \frac{1}{\sqrt{2k+2l}}(\sqrt{2k}b_1^\alpha + \sqrt{2l}b_2^\alpha) = \frac{1}{\sqrt{2k+2l}}(a_1^\alpha - a_2^\alpha + \dots - a_{2k+2l}^\alpha). \quad (\text{A4})$$

The corresponding equations (38) take the form

$$\text{str}(\sigma) = -\text{str}\left(\left([b_1^0, b_1^1] - 1\right)\sigma\right) \quad (\text{A5})$$

and

$$\text{str}(\sigma) = -\text{str}\left(\left([b_2^0, b_2^1] - 1\right)\sigma\right). \quad (\text{A6})$$

Using the following simple identity which holds for any trace on S_N ,

$$\text{str}\left(\frac{1}{2k} \sum_{p=1}^{2k} \sum_{q=2k+1}^{2k+2l} K_{pq}\sigma\right) = 2l \text{str}(K_{1(2k+1)}\sigma)$$

one can rewrite the right hand side of (A5) as

$$\text{str}\left(\left([b_1^0, b_1^1] - 1\right)\sigma\right) = \text{str}\left(\left([b_1^0, b_1^1] - 1 - \frac{1}{2k} \sum_{p=1}^{2k} \sum_{q=2k+1}^{2k+2l} K_{pq}\right)\sigma\right) + 2l \text{str}(K_{1(2k+1)}\sigma). \quad (\text{A7})$$

The direct analysis based on the commutation relations (2) and (3) then shows that the first term on the right hand side of (A7) is the supertrace of a linear combination of permutations which all contain the cycle c_2 in their decompositions. The second term is the supertrace of the permutation which contains the cycle c (A3) in its decomposition. It is easy to see that for each of these terms the number of cycles of even length is $E(\sigma) - 1$. This allows us to apply the equation (38) to each of this terms due to the inductive hypothesis. We identify c_0 with c_2 and c , respectively, for the first and second terms on the right hand side of (A7). As a result the equation (A5) turns out to be transformed to the form

$$\begin{aligned} \text{str}(\sigma) &= \text{str}\left(\left([b_2^0, b_2^1] - 1\right)\left([b_1^0, b_1^1] - 1 - \frac{1}{2k} \sum_{p=1}^{2k} \sum_{q=2k+1}^{2k+2l} K_{pq}\right)\sigma\right) \\ &+ \text{str}\left(\left([b^0, b^1] - 1\right)2lK_{1(2k+1)}\sigma\right), \end{aligned} \quad (\text{A8})$$

Analogously one obtains for (A6)

$$\begin{aligned} \text{str}(\sigma) &= \text{str}\left(\left([b_1^0, b_1^1] - 1\right)\left([b_2^0, b_2^1] - 1 - \frac{1}{2l} \sum_{p=1}^{2k} \sum_{q=2k+1}^{2k+2l} K_{pq}\right)\sigma\right) \\ &+ \text{str}\left(\left([b^0, b^1] - 1\right)2kK_{1(2k+1)}\sigma\right). \end{aligned} \quad (\text{A9})$$

Let us prove that the difference of the right hand sides of (A8) and (A9) vanishes. With the aid of the simple consequence of the S_N invariance

$$\frac{1}{4kl} \text{str} \left([b_i^0, b_i^1] \sum_{p=1}^{2k} \sum_{q=2k+1}^{2k+2l} K_{pq} \sigma \right) = \text{str} \left([b_i^0, b_i^1] K_{1(2k+1)} \sigma \right), \quad i = 1, 2$$

this difference can be transformed to the form

$$X = \text{str} \left(\left(2k [b_1^0, b_1^1] - 2l [b_2^0, b_2^1] \right) K_{1(2k+1)} \sigma \right), \quad (\text{A10})$$

where we have taken into account that

$$\text{str} \left([b^0, b^1] K_{1(2k+1)} \sigma \right) = 0 \quad (\text{A11})$$

as a consequence of the inductive hypothesis and GLC (37) and that

$$\text{str} \left(\left([b_2^0, b_2^1] - 1 \right), \left([b_1^0, b_1^1] - 1 \right) \right) \sigma = 0 \quad (\text{A12})$$

since each term in the commutator belongs to the group algebra of S_N and commutes with σ so that (A12) vanishes for any supertrace on the group algebra of S_N .

Using the relation $b_1^\alpha = 1/\sqrt{2k} \left(\sqrt{2k+2l} b^\alpha - \sqrt{2l} b_2^\alpha \right)$ one transforms X to the form

$$X = 2 \text{str} \left(\left((k+l) [b^0, b^1] - \sqrt{l(k+l)} [b^0, b_2^1] - \sqrt{l(k+l)} [b_2^0, b^1] \right) K_{1(2k+1)} \sigma \right). \quad (\text{A13})$$

Due to the S_N invariance the second term on the right hand side of (A13) can be rewritten as

$$\begin{aligned} -2\sqrt{l(k+l)} \text{str} \left([b^0, b_2^1] K_{1(2k+1)} \sigma \right) &= -\frac{\sqrt{l(k+l)}}{k+l} \sum_{p=1}^{2k+2l} \text{str} \left(c^p [b^0, b_2^1] c^{-p} K_{1(2k+1)} \sigma \right) \\ &= -\frac{\sqrt{l}}{\sqrt{k+l}} \text{str} \left(\left[b^0, \sum_{p=1}^{2k+2l} (-1)^p c^p b_2^1 c^{-p} \right] K_{1(2k+1)} \sigma \right) = -2l \text{str} ([b^0, b^1] K_{1(2k+1)} \sigma). \end{aligned} \quad (\text{A14})$$

Analogously one can transform the third term on the right hand side of (A13). Using again (A11) one concludes that $X = 0$.

Thus it is shown that the number of solutions of (39) is equal to the number of the conjugacy classes in S_N with $E(\sigma) = 0$. This completes the proof of *Theorem 1*.

B Consistency for $\lambda \neq -1$

Let us prove by induction that the equations (55) are true for any two $\mu_1 \equiv \mu_{K_1}$ and $\mu_2 \equiv \mu_{K_2}$ such that both $\lambda_1 \equiv \lambda_{K_1} \neq -1$ and $\lambda_2 \equiv \lambda_{K_2} \neq -1$. To implement induction one selects from (47) a part of order k in μ and observes that it contains the anticommutator of b^L with a degree k polynomial in b^M while the part on the right hand side of the differential version (53) of (47) which is of the same order in μ has the order $k-1$ as the polynomial of b^M . This happens because of the presence of the commutator $[b^L, S]$

which is a degree zero polynomial due to the basic relations (2), (3). As a result, the inductive hypothesis allows one to use the properties of the supertrace provided that the above commutator is always handled as the right hand side of (2) (*i.e.* it is not allowed to represent it again as a difference of the second-order polynomials).

Direct differentiation with the aid of (51) gives

$$\begin{aligned}
(1 + \lambda_2) \frac{\partial}{\partial \mu_2} \int (\lambda_1 t_1 - t_2) \text{str} \left(e^{t_1 S} [b^1, S] e^{t_2 S} \sigma \right) D^1 t - (1 \leftrightarrow 2) &= \\
\left(\int (1 + \lambda_2) (\lambda_1 t_1 - t_2) \text{str} \left(e^{t_1 S} [b^1, b^2] e^{t_2 S} \sigma \right) D^1 t - (1 \leftrightarrow 2) \right) &+ \\
\left(\int (1 + \lambda_2) (\lambda_1 (t_1 + t_2) - t_3) \text{str} \left(e^{t_1 S} b^2 e^{t_2 S} [b^1, S] e^{t_3 S} \right) D^2 t - (1 \leftrightarrow 2) \right) &+ \\
\left(\int (1 + \lambda_2) (\lambda_1 t_1 - t_2 - t_3) \text{str} \left(e^{t_1 S} [b^1, S] e^{t_2 S} b^2 e^{t_3 S} \sigma \right) D^2 t - (1 \leftrightarrow 2) \right). & \quad (\text{B1})
\end{aligned}$$

We have to show that the right hand side of (B1) vanishes. Let us first transform the second and the third terms on the right-hand side of (B1). The idea is to move the operators b^2 through the exponentials towards the commutator $[b^1, S]$ so that to use then Jacobi identities for the double commutators. This can be done in two different ways inside the supertrace so that one has to fix appropriate weight factors for each of these processes. The correct weights turn out to be

$$\begin{aligned}
D^2 t (\lambda_1 (t_1 + t_2) - t_3) b^2 &\equiv D^2 t (\lambda_1 - t_3 (1 + \lambda_1)) b^2 = \\
D^2 t \left(\left(\frac{\lambda_1 \lambda_2}{1 + \lambda_2} - t_3 (1 + \lambda_1) \right) \overrightarrow{b^2} + \frac{\lambda_1}{1 + \lambda_2} \overleftarrow{b^2} \right) & \quad (\text{B2})
\end{aligned}$$

and

$$\begin{aligned}
D^2 t (\lambda_1 t_1 - t_2 - t_3) b^2 &\equiv D^2 t ((\lambda_1 + 1) t_1 - 1) b^2 = \\
D^2 t \left(\left(t_1 (1 + \lambda_1) - \frac{1}{1 + \lambda_2} \right) \overleftarrow{b^2} - \frac{\lambda_2}{1 + \lambda_2} \overrightarrow{b^2} \right) & \quad (\text{B3})
\end{aligned}$$

in the second and third terms on the right hand side of (B1), respectively. Here the notations \overrightarrow{A} and \overleftarrow{A} imply that the operator A has to be moved from its position to the right and to the left, respectively. Using (52) along with the simple formula

$$\int \phi(t_3, \dots, t_{n+1}) D^n t = \int t_1 \phi(t_2, \dots, t_n) D^{n-1} t \quad (\text{B4})$$

one finds that all terms which involve both $[b^1, S]$ and $[b^2, S]$ cancel pairwise after antisymmetrization $1 \leftrightarrow 2$.

As a result, one is left with some terms involving double commutators which by virtue of Jacobi identities and antisymmetrization all reduce to

$$\int \left(\lambda_1 \lambda_2 t_1 + t_2 - t_1 t_2 (1 + \lambda_1) (1 + \lambda_2) \right) \text{str} \left(\exp(t_1 S) [S, [b^1, b^2]] \exp(t_2 S) \sigma \right) D^1 t. \quad (\text{B5})$$

Finally one observes that this expression can be equivalently rewritten in the form

$$\int \left(\lambda_1 \lambda_2 t_1 + t_2 - t_1 t_2 (1 + \lambda_1)(1 + \lambda_2) \right) \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right) \text{str} \left(\exp(t_1 S) [b^1, b^2] \exp(t_2 S) \sigma \right) D^1 t \quad (\text{B6})$$

and after integration by parts cancel the first term on the right-hand side of (B1). Thus it is shown that the equations (53) are mutually compatible for the case $\lambda_{1,2} \neq -1$.

Analogously one can show that the equations (53) are consistent with (54). Actually, let $\lambda_1 = -1$, $\lambda_2 \neq -1$. Let us prove that

$$\frac{\partial}{\partial \mu_2} \text{str}([b^1, \exp(S)] \sigma) = 0 \quad (\text{B7})$$

provided that the supertrace is well defined for the lower order polynomials. The explicit differentiation gives

$$\begin{aligned} \frac{\partial}{\partial \mu_2} \text{str}([b^1, \exp(S)] \sigma) &= \int \text{str}([b^1, \exp(t_1 S) b^2 \exp(t_2 S)] \sigma) D^1 t \\ &= (1 + \lambda_2)^{-1} \text{str}([b^1, (b^2 \exp(S) + \lambda_2 \exp(S) b^2)] \sigma) + \dots \end{aligned} \quad (\text{B8})$$

where dots denote some terms of the form $\text{str}([b^1, B] \sigma)$ involving further commutators inside B , which therefore amount to some lower order polynomials and vanish by the inductive hypothesis. As a result, one finds

$$\begin{aligned} \frac{\partial}{\partial \mu_2} \text{str}([b^1, \exp(S)] \sigma) &= (1 + \lambda_2)^{-1} \text{str}((b^2 [b^1, \exp(S)] + \lambda_2 [b^1, \exp(S)] b^2) \sigma) \\ &\quad + (1 + \lambda_2)^{-1} \text{str}([b^1, b^2] \exp(S) + \lambda_2 \exp(S) [b^1, b^2]) \sigma, \end{aligned} \quad (\text{B9})$$

which expression vanishes by the inductive hypothesis too.

C Consistency for $\lambda = -1$

In order to prove (70) we use the inductive hypothesis **(i)**. In this appendix we use the convention that any upper or lower indices denoted by the same letter are automatically symmetrized, *e.g.* $F^{II} \stackrel{\text{def}}{=} \frac{1}{2}(F^{I_1 I_2} + F^{I_2 I_1})$. Let us write the identity

$$0 = \sum_M \text{str}([\exp(S') \{b^I, f^{IM}\} \mu_M, b^J b^J] \sigma) - (I \leftrightarrow J) \quad (\text{C1})$$

which holds due to *Lemma 5* for all terms of degree $k - 1$ in μ with $E(\sigma) \leq l + 1$ and for all lower order polynomials in μ (one can always move f^{IJ} to σ combining them into a combination of elements of S_N analyzed in *Lemma 5*).

The straightforward calculation of the commutator on the right-hand-side of (C1) gives $0 = X_1 + X_2 + X_3$, where

$$\begin{aligned} X_1 &= - \sum_{M,L} \int \text{str} \left(\exp(t_1 S') \{b^J, F^{JL}\} \mu_L \exp(t_2 S') \{b^I, f^{IM}\} \mu_M \sigma \right) D^1 t - (I \leftrightarrow J), \\ X_2 &= \sum_M \text{str} \left(\exp(S') \{ \{b^J, F^{IJ}\}, f^{IM} \} \mu_M \sigma \right) - (I \leftrightarrow J), \\ X_3 &= \sum_M \text{str} \left(\exp(S') \{ b^I, \{ b^J, [f^{IM}, b^J] \} \} \mu_M \sigma \right) - (I \leftrightarrow J). \end{aligned} \quad (\text{C2})$$

The terms bilinear in f in X_1 cancel due to the antisymmetrization ($I \leftrightarrow J$) and the inductive hypothesis (i). As a result, one can transform X_1 to the form

$$X_1 = \left(-\frac{1}{2} [L^{JJ}, R^{II}] + 2 \text{str} \left(e^{S'} \{b^I, f^{IJ}\} \mu^J \sigma \right) \right) - (I \leftrightarrow J). \quad (\text{C3})$$

Substituting $F^{IJ} = \mathcal{C}^{IJ} + \nu f^{IJ}$ and $f^{IM} = \nu^{-1}([b^I, b^M] - \mathcal{C}^{IM})$ one transforms X_2 to the form

$$X_2 = 2\mathcal{C}^{IJ} R^{IJ} - 2 \left(\text{str} \left(e^{S'} \{b^J, f^{IJ}\} \mu^I \sigma \right) - (I \leftrightarrow J) \right) + Y, \quad (\text{C4})$$

where

$$Y = \text{str} \left(e^{S'} \{ \{b^J, f^{IJ}\}, [b^I, S'] \} \sigma \right) - (I \leftrightarrow J). \quad (\text{C5})$$

Using that

$$\text{str} \left(\exp(S') [P f^{IJ} Q, S'] \sigma \right) = 0 \quad (\text{C6})$$

provided that the inductive hypothesis can be used, one transforms Y to the form

$$Y = \text{str} \left(e^{S'} \left(\begin{aligned} & - [f^{IJ}, (b^I S' b^J + b^J S' b^I)] - b^I [f^{IJ}, S'] b^J - b^J [f^{IJ}, S'] b^I \\ & + [f^{IJ}, \{b^I, b^J\}] S' \end{aligned} \right) \sigma \right). \quad (\text{C7})$$

Let us rewrite X_3 in the form $X_3 = X_3^s + X_3^a$ where

$$\begin{aligned} X_3^s &= \frac{1}{2} \sum_M \text{str} \left(e^{S'} \left(\{b^I, \{b^J, [f^{IM}, b^J]\}\} + \{b^J, \{b^I, [f^{IM}, b^J]\}\} \right) \mu_M \sigma \right) - (I \leftrightarrow J), \\ X_3^a &= \frac{1}{2} \sum_M \text{str} \left(e^{S'} \left(\{b^I, \{b^J, [f^{IM}, b^J]\}\} - \{b^J, \{b^I, [f^{IM}, b^J]\}\} \right) \mu_M \sigma \right) - (I \leftrightarrow J). \end{aligned}$$

With the aid of the Jacobi identities $[f^{IM}, b^J] - [f^{JM}, b^I] = [f^{IJ}, b^M]$ one expresses X_3^s in the form

$$X_3^s = \frac{1}{2} \text{str} \left(e^{S'} \left(\{b^I, b^J\} [f^{IJ}, S'] + [f^{IJ}, S'] \{b^I, b^J\} + 2b^I [f^{IJ}, S'] b^J + 2b^J [f^{IJ}, S'] b^I \right) \sigma \right).$$

X_3^a can be transformed to the form

$$X_3^a = \frac{1}{2} \sum_M \text{str} \left(e^{S'} [F^{IJ}, [f^{IM}, b^J]] \mu_M \sigma \right) - (I \leftrightarrow J). \quad (\text{C8})$$

By virtue of the substitutions $F^{IJ} = \mathcal{C}^{IJ} + \nu f^{IJ}$ and $f^{IM} = \nu^{-1}([b^I, b^M] - \mathcal{C}^{IM})$ in (C8) one finds after simple transformations that $Y + X_3 = 0$. From (C3) and (C4) it follows then that the right hand side of (C1) equals to $\frac{1}{2}([L^{II}, R^{JJ}] - [L^{JJ}, R^{II}]) + 2\mathcal{C}^{IJ} R^{IJ}$. This completes the proof of the consistency conditions (70).

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